Field-theoretic description of the multicritical behavior of systems with two order parameters

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A field-theoretic description of phase transformations in complex systems with two interacting order parameters is given. For three-dimensional systems in the two-loop approximation a renormalization-group analysis of the scaling functions is carried out directly, and the fixed points corresponding to stability of the bicritical and tetracritical behavior are identified. The critical exponents at the multicritical points in the two-loop approximation are calculated with the use of the Padé–Borel summation technique.
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There exists a wide class of systems in which the observed phase transition cannot be described by one order parameter transforming according to one irreducible representation. Magnetic crystals whose magnetic structure is described by two or more irreducible representations (the antiferromagnets Cr_2TeO_6, $KCuF_3$, $GdAlO_3$, MnF_2, and others) afford exceptionally many such examples. Structural phase transitions whose description requires several order parameters have been found in $KMnF_3$, boracites, and other substances. The phase diagrams of such subsystems have a special multicritical point, which is of a bicritical or tetracritical character.[1,2] In the first case, two lines of second-order phase transitions and one line of a first-order phase transition intersect, and in the second case four lines of second-order phase transitions intersect. Close to a multicritical point the system demonstrates a specific critical behavior characterized by the competition between the different types of ordering. If bicritical behavior is realized, one order parameter in the system displaces the other, while tetracritical behavior admits the existence of a mixed phase with both types of ordering coexisting.

The model Hamiltonian of a system with two coupled order parameters ϕ and ψ, belonging to two different irreducible representations of dimension n and m, has the form

$$H_0=\int d^dX\left(\frac{1}{2}[r_1\phi^2+r_2\psi^2+(\nabla\phi)^2+(\nabla\psi)^2]+\frac{u_{10}}{4!}(\phi^2)^2+\frac{u_{20}}{4!}(\psi^2)^2+\frac{2u_{30}}{4!}\phi^2\psi^2\right), \qquad (1)$$

TABLE I. Values of the fixed points and eigenvalues.

n	m	u_1^*	u_2^*	u_3^*	b_1	b_2	b_3
1	1	0.93498196	0.93498196	0.93498196	0.090410	0.523089	0.667315
		1.06446157	1.06446157	0.00000000	0.653550	-0.169273	0.653550
		0.53223078	0.53223078	1.59669235	-0.205852	0.008999	0.653550
1	2	0.87048304	0.84551387	0.76415961	0.505216	0.681095	0.007546
		1.06446157	0.93498196	0.00000000	0.653550	-0.085636	0.667315
		0.82961991	0.82961991	0.82961991	-0.008388	0.474448	0.681378
1	3	1.05549842	0.82655711	0.13586107	0.016030	0.681066	0.649077
		1.06446157	0.82961991	0.00000000	0.653550	-0.019503	0.681379
		0.74314276	0.74314276	0.74314276	-0.093023	0.432386	0.695090
1	4	1.06446157	0.74314276	0.00000000	0.653550	0.695090	0.033451
		0.67140562	0.67140562	0.67140562	-0.165868	0.396007	0.708159
1	5	1.06446157	0.67140562	0.00000000	0.653550	0.076394	0.708159
		0.61123804	0.61123804	0.61123804	-0.228952	0.364435	0.720466
		0.96224764	0.65141432	-0.32221693	0.589565	-0.109812	0.728987
2	2	0.93494419	0.93494419	0.01059832	0.667270	0.667335	0.001647
		0.93498195	0.93498195	0.00000000	0.667315	-0.001673	0.667315
		0.74314276	0.74314276	0.74314276	-0.093023	0.432386	0.695090
2	3	0.93498195	0.82961991	0.00000000	0.667315	0.064781	0.681379
		0.67140562	0.67140562	0.67140562	-0.165868	0.396007	0.708159
2	4	0.93498195	0.74314276	0.00000000	0.667315	0.118008	0.695090
		0.61123804	0.61123804	0.61123804	-0.228952	0.364435	0.720466
		0.81776815	0.68640603	-0.37358192	0.555319	-0.212087	0.755115
3	3	0.82961991	0.82961991	0.00000000	0.681379	0.681379	0.131538
		0.73717819	0.73717819	-0.38320676	-0.208150	0.766734	0.545585
		0.61123804	0.61123804	0.61123804	-0.228952	0.720466	0.364435

$$\phi^2=\sum_{i=1}^{n}\phi_i^2,\quad \psi^2=\sum_{i=1}^{m}\psi_i^2,\quad (\nabla\phi)^2=\sum_{i=1}^{n}(\nabla\phi_i)^2,\quad (\nabla\psi)^2=\sum_{i=1}^{m}(\nabla\psi_i)^2.$$

The problem of a phase transition in such a system was analyzed in Ref. 3 and independently in Ref. 4. The model under consideration was investigated using Wilson's renormalization-group technique on the basis of the ϵ expansion method in the one-loop approximation. In Ref. 3, an attempt was made to trace the dependence of the character of the multicritical behavior on the numbers n and m. However, numerous investigations of systems characterized by one order parameter performed in the last few years show that the predictions made in the one-loop approximation, especially on the basis of the ϵ expansion, can differ strongly from the real critical behavior. To shed light on this question with respect to multicritical phenomena and to determine more accurately the dependence of the character of the multicritical behavior on the structure of the order parameters, we have constructed a field-theoretic description of the system (1) in the two-loop approximation. We employed the mass theory of Ref. 5, which makes it possible to describe three-dimensional systems directly without using an ϵ expansion ($\epsilon = 4-d$, where d is the dimensionality of the system). Investigations of critical phenomena show[6] that this approach gives the best description of the critical behavior, and its application together with methods for summing asymptotically convergent series make it possible to achieve high accuracy.

As is well known, in the field-theoretic approach[7] the asymptotic critical behavior and structure of phase diagrams in the fluctuation region are determined by the Callan–Symanzik renormalization-group equation for the vertex parts of the irreducible Green's functions. To calculate the β functions and the critical exponents as functions of the renormalized interaction vertices u_1, u_2, and u_3 (scaling functions) appearing in the renormalization-group equation, we used the standard method based on the Feynmann diagram technique and the renormalization procedure.[6] As a result, we obtain for the β functions in the two-loop approximation

$$\beta_1(u_1,u_2,u_3) = -u_1 + \frac{(n+8)}{6}u_1^2 + \frac{m}{6}u_3^2 - \frac{(41n+190)}{243}u_1^3 - \frac{2m}{27}u_3^3 - \frac{23m}{243}u_1u_3^2,$$

$$\beta_2(u_1,u_2,u_3) = -u_2 + \frac{(m+8)}{6}u_2^2 + \frac{n}{6}u_3^2 - \frac{(41m+190)}{243}u_2^3 - \frac{2n}{27}u_3^3 - \frac{23n}{243}u_2u_3^2,$$

$$\beta_3(u_1,u_2,u_3) = -u_3 + \frac{2}{3}u_3^2 + \frac{(n+2)}{6}u_1u_3 + \frac{(m+2)}{6}u_2u_3 - \frac{5n+5m+72}{486}u_3^3 - \frac{23(n+2)}{486}u_1^2u_3 - \frac{23(m+2)}{486}u_2^2u_3 - \frac{(n+2)}{9}u_1u_3^2 - \frac{(m+2)}{9}u_2u_3^2, \tag{2}$$

and for the scaling functions η and γ

$$\eta_1(u_1,u_2,u_3) = \frac{2}{243}((n+2)u_1^2 + mu_3^2), \quad \eta_2(u_1,u_2,u_3) = \frac{2}{243}((m+2)u_2^2 + nu_3^2); \tag{3}$$

$$\gamma_\phi^2(u_1,u_2,u_3) = \frac{n+2}{6}u_1 + \frac{m}{6}u_3 - \frac{n+2}{18}u_1^2 - \frac{m(n-m+2)}{36}u_3^2 + \frac{m(n+2)}{36}u_1u_3 - \frac{m(m+2)}{36}u_2u_3,$$

$$\gamma_\psi^2(u_1,u_2,u_3)=\frac{m+2}{6}u_2+\frac{n}{6}u_3-\frac{m+2}{18}u_2^2-\frac{n(m-n+2)}{36}u_3^2$$
$$+\frac{n(m+2)}{36}u_2u_3-\frac{n(n+2)}{36}u_1u_3. \quad (4)$$

It is well known that perturbation series are asymptotically convergent, and the interaction vertices of the fluctuations of the order parameters in the fluctuation region $r_1=r_2\to 0$ are large enough so that expressions (2)–(4) can be used directly. For this reason, to extract the required physical information from the expressions obtained, we employed the generalized Padé–Borel method for summing asymptotically convergent series. The direct and inverse Borel transformations extended to the multiparameter case and preserving the symmetry of the system have the form

$$f(u_1,u_2,u_3)=\sum_{i,j,k} c_{ijk}u_1^iu_2^ju_3^k=\int_0^\infty e^{-1}F(u_1t,u_2t,u_3t)dt,$$

$$F(u_1,u_2,u_3)=\sum_{i,j,k}\frac{c_{ijk}}{(i+j+k)!}u_1^iu_2^ju_3^k. \quad (5)$$

A series in the auxiliary variable λ is introduced for analytical continuation of the Borel transform of the function:

$$\tilde{F}(u_1,u_2,u_3,\lambda)=\sum_{k=0}^{\infty}\lambda^k\sum_{i=0}^{k}\sum_{j=0}^{k-i}\frac{c_{i,j,k,k-i-j}}{k!}u_1^iu_2^ju_3^{k-i-j}, \quad (6)$$

to which the $[L/M]$ Padé approximation is applied at the point $\lambda=1$. We used the [2/1] approximant to calculate the β functions in the two-loop approximation. The nature of the multicritical behavior is determined by the existence of a stable fixed point satisfying the system of equations

$$\beta_k(u_1^*,u_2^*,u_3^*)=0 \quad (k=1,2,3). \quad (7)$$

The values obtained for the fixed points by solving the system (7) for the most interesting values of the number of components n and m of the order parameters are presented in Table I. The requirement that the fixed point be stable reduces to the condition that the eigenvalues b_1, b_2, and b_3 (see Table I) of the matrix

$$B_{ij}=\frac{\partial\beta_i(u_1^*,u_2^*,u_3^*)}{\partial u_j} \quad (8)$$

lie in the right-hand complex half plane.

In Ref. 3, three types of stable fixed points corresponding to different values of n and m were found. The regions of existence of these fixed points in the (n,m) plane in the one-loop approximation are shown in Fig. 1a. The type 1 corresponds to an isotropic fixed point, where $u_1^*=u_2^*=u_3^*$ and the Hamiltonian (1) is effectively the same as the Hamiltonian of a system with one $(n+m)$-component order parameter with the complete symmetry $SO(n+m)$ higher than the $SO(n)\times SO(m)$ symmetry of the initial system (manifestation of the fluctuation-determined asymptotic symmetry at the multicritical point). For points of the type 2, all three values of u_i^* are nonzero and could be different

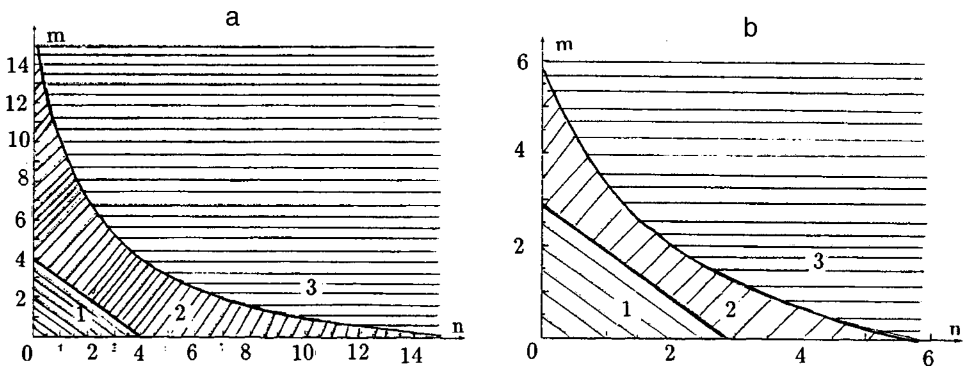

FIG. 1. Regions of stability of fixed points determined: a) in Refs. 3 and 4 in the first order of the ϵ expansion, and b) in the present letter in the field-theoretic approach in the two-loop approximation with $d=3$.

from one another. They correspond to the lowest symmetry $SO(n)\times SO(m)$ of the initial Hamiltonian. The type-3 points correspond to decoupled order parameters, since at these points $u_3^*=0$. They also correspond to the higher symmetry $SO(n)\oplus SO(m)$. Figure 1b shows the regions of existence of different types of fixed points, which we obtained in the two-loop approximation without using the ϵ expansion. The boundary of the region of stability of an isotropic fixed point now passes along the straight line $n+m=2.9088$, i.e., the highest asymptotic symmetry of the system is $SO(2)$, and the region of existence of type-2 points has become so narrow that it contains only five points of physical interest. The large change in the picture indicates that the correspondence between the one-loop approximation and the real multicritical behavior is weak.

The phase diagrams for the Hamiltonian (1) in the mean-field approximation (neglecting fluctuations) are well known.[3,8] In the case $u_3^2<u_1u_2$, a tetracritical point is realized, and therefore a mixed phase with $\phi\neq 0$ and $\psi\neq 0$ can exist. In the opposite case, $u_3^2\geqslant u_1u_2$, the phase diagram possesses a bicritical point and a mixed phase does not arise. However, as shown in Ref. 3, taking account of fluctuations can greatly change the character of the phase diagram in the critical region. For this, the phase portrait of the system must be investigated by solving the system of equations ($r=r_1=r_2$)

$$r\frac{\partial u_k}{\partial r}=\beta_k(u_1^*,u_2^*,u_3^*), \tag{9}$$

which gives the phase trajectories in the space of the vertices u_k. In the limit $r\to 0$, depending on the initial values $u_k^{(0)}$ of the vertices, the phase trajectories either leave the region of stability of the Hamiltonian (1) with the realization of a first-order phase transition or arrive at a stable fixed point from the set of fixed points examined above with a definite symmetry of the system. The phase trajectories can cross regions where the condition of tetra- or bicriticality holds for the vertices. As a result, inclusions of curves of first-order phase transitions and a set of critical points appear in the critical region in the phase diagrams corresponding to the mean-field theory.[3]

The more accurate determination made in the present letter of the values and type of the stable fixed points does not materially change the analysis of the phase portrait and phase diagrams performed in Ref. 3 for the case $n=m=1$. However, in the cases with

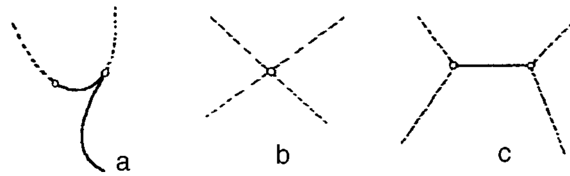

FIG. 2. Possible types of phase diagrams. The solid lines correspond to curves of first-order phase transitions and the dashed lines correspond to second-order transitions.

$n+m>2.9088$ the large change found in the values of the fixed points and their stability conditions should appreciably change the phase diagrams in the critical region and should lead to other forms of symmetry of the system at the multicritical point. As one can see from the values of u_k^* presented in Table I, the stable fixed points for $n+m\geqslant 3$ are tetracritical, i.e., $u_3^2<u_1u_2$. For this reason, if the initial values of the vertices satisfy the bicriticality condition, then the phase trajectories will always leave the region of stability of the Hamiltonian (1) and the phase diagrams that are bicritical outside the critical region will contain inclusions of curves of first-order phase transitions in the critical region (Fig. 2a). However, if the initial values of the vertices satisfy the tetracriticality condition, then for type-3 fixed points the phase diagrams are of a tetracritical character both outside and inside the critical region (Fig. 2b). The computed values of the critical exponents characterizing the tetracritical behavior are presented in Table II. The two sets of exponents for systems with $n\neq m$ reflect the fact that the critical behavior of the two

TABLE II. Values of the critical exponents.

n	m	i	η	ν	α	β	γ
1	1	-	0.02878	0.67371	-0.02114	0.34655	1.32804
1	2	1	0.02832	0.70402	-0.11206	0.36198	1.38810
		2	0.02834	0.70461	-0.11383	0.36229	1.38925
1	3	1	0.02796	0.66129	0.01613	0.33989	1.30409
		2	0.02827	0.71516	-0.14547	0.36769	1.41010
1	4	1	0.02798	0.63796	0.08613	0.32790	1.25807
		2	0.02727	0.73165	-0.19495	0.37580	1.44335
1	5	1	0.02798	0.63796	0.08613	0.32790	1.25807
		2	0.02597	0.75502	-0.26505	0.38731	1.49042
2	2	-	0.02878	0.67521	-0.02563	0.34732	1.33099
2	3	1	0.02878	0.67371	-0.02114	0.34655	1.32804
		2	0.02832	0.70474	-0.11423	0.36235	1.38953
2	4	1	0.02878	0.67371	-0.02114	0.34655	1.32804
		2	0.02727	0.73165	-0.19495	0.37580	1.44335
3	3	-	0.04532	0.93429	-0.80288	0.48832	1.82625

different order parameters can be determined independently in different experiments. However, the critical behavior of the specific heat of a system is determined by the large exponent. For type-2 fixed points, analysis of the phase portrait of the system shows that phase diagrams of two kinds are possible: They are tetracritical outside the critical region, and within the critical region diagrams of the first kind have a tetracritical point (Fig. 2b) while diagrams of the second kind contain inclusions of a curve of a first-order phase transition with two bicritical points (Fig. 2c).

In closing, we hope that the characteristic features revealed here of the multicritical behavior will find applications in the analysis of experimental works on the multicritical behavior of systems with competing order parameters.

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